

Nonequilibrium variational principle for the time evolution of an ionized gas

F. Vázquez

Departamento de Física, Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Avenida Universidad 1001, 62210 Cuernavaca, Morelos, México

J. A. del Río

Laboratorio de Energía Solar, IIM, Universidad Nacional Autónoma de México, AP 34, 62580 Temixco, Morelos, México

(Received 21 May 1991; revised manuscript received 3 September 1992)

A restricted variational formulation for extended irreversible thermodynamics is proposed. Then, it is applied to a heat- and electricity-conducting viscous fluid, and the resulting equations are reduced to order two to be compared with those well known of kinetic theory. A better understanding of the phenomenological coefficients than in previous reported works is obtained.

PACS number(s): 05.70.Ln, 03.40.Gc, 51.50.+v, 47.65.+a

I. INTRODUCTION

Variational principles to study the movement of fluids have been used for a variety of cases: perfect fluids, conducting fluids, non-Newtonian fluids, etc. The applications of variational principles of the Hamilton type have been mostly restricted to perfect fluids (Kelvin [1] provided one in the Lagrangian form) since their success has been limited in other areas of fluid mechanics. The steady state without flow [2] and a more general case including flow [3] have been formulated from the classical variational point of view in the Eulerian form for conducting perfect fluids in the presence of an electromagnetic field of forces and the viscous conducting fluid has been treated in a channel flow [4]. The nonsteady problem for the magnetohydrodynamics of perfect fluids has been worked out in both the Eulerian and Lagrangian forms from classical principles [5]. The constraints on the system are introduced in the above formalisms by using Lagrange multipliers.

When the equations include dissipative effects, the presence of non-self-adjoint operators has induced the search of a restricted type of variational principles (introduced by Onsager [6]) since the classical ones are not applicable. The description of the relaxation of magnetically confined plasmas based on local potential variational principles derived from the minimum entropy production principle was published a long time ago [7–9]. Recently, Hameiri and Bhattacharjee [10] applied Prigogine's principle to obtain the equation for the electric charge flux in an incompressible conducting fluid in the stationary state. They showed that the fluid evolves to a relaxed state of minimum entropy production rather than to a state of minimum energy, such as Taylor's theory assumes [11]. Later, Rasband, Mason, and Matheson [12] found the time evolution equations for the conserved field variables describing the system through the so-called generalized entropy production (GEP) functional. Their work is based on a variational principle which appears as a generalization of Prigogine's principle. Recently, Lebon and Dauby [13] formulated a varia-

tional principle within an extended thermodynamic version for the heat-wave propagation in dielectric crystals at low temperatures, deriving the Guyer-Krumhansl equation.

As is well known, there is a wide interest in applying variational principles to the recent theories of extended thermodynamics and some other related works can be found in the literature. See, for example, Bhattacharya [14] within the wave approach of thermodynamics, Nettleton [15] with a Lagrangian formulation of extended thermodynamics using similar thermodynamical variables as those of Onsager [6], Sieniutycz [16–18] with a relaxation factor in a functional giving hyperbolic equations for extended thermodynamics, and Eu [19] within a variational scheme on the generalized equations of state.

In this paper we apply a recently sketched variational formalism [20] of the restricted type to a heat- and electricity-conducting fluid. This principle was formulated in the framework of extended irreversible thermodynamics [21] (EIT) as presented in Ref. [21]. Although we share the same objectives, our principle has some essential differences with respect to the above-mentioned ones since it is developed within a distinctive theoretical framework. We will make some comparative comments later on. With the aid of the restricted principle for EIT we have arrived at the time evolution equations to second order of nonconserved variables of the conducting fluid, for which viscous properties are supposed, under an external electromagnetic field of forces. According to the formalism, for times smaller than the relaxation times, the fluid evolves in such a way that a functional is an invariant extreme value in front of a set of transformations proposed for the extended thermodynamic space of variables. The set of equations for the conserved variables is closed with equations of the Maxwell-Cattaneo-Vernotte type which are derived from the variational principle. Thus we obtain a variational description of transient transport processes for the system. The advantages are, as is well known, twofold: to summarize the subject suggesting analogies and generalizations, and to inspire methods for obtaining model solutions to the problem in-

cluding the use of additional information such as intuitive considerations [22].

The conducting fluid has also been studied in a standard EIT by Goldstein and García-Colín [23] and from a microscopic point of view by Yang [24] and Spitzer [25]. Jou, Ferrer, and Llebot [26] have studied the stability of the steady state under an external electric field without considering the cross effects in the conducting fluid.

In Sec. II the extended irreversible thermodynamics we are using is described briefly. Next, in Sec. III we state the variational formulation of the restricted type for EIT, based on a functional defined in terms of the balance of the generalized thermodynamic potential η . Then the variational principle is applied to a heat- and electricity-conducting viscous fluid for which we choose a suitable extended thermodynamic space and we comment on the results. The paper is closed with some concluding remarks in Sec. IV.

II. EXTENDED IRREVERSIBLE THERMODYNAMICS

In its axiomatic form EIT can be stated as follows [27]. The space of the thermodynamic variables is enlarged with respect to that of linear irreversible thermodynamics in order to include the nonconserved variables (fluxes of the system). Thus to describe the system off local equilibrium it is assumed that a generalized thermodynamic potential η (entropylike function) exists and is a well-behaved local function of all the variables of the extended thermodynamics space [28,29]; its time evolution is determined by an equation of the Gibbs form which must reduce to the usual Gibbs equation in local equilibrium. (ii) The generalized thermodynamic potential η obeys a balance equation of the form

$$\rho \frac{d\eta}{dt} = -\nabla \cdot \mathbf{J}_\eta + \sigma \quad (1)$$

with \mathbf{J}_η the flux of the entropylike function η and σ the production term, which has the properties remarked by Rodríguez and López de Haro [30]. Concerning the methodology the following is assumed: (iii) The coefficients appearing in the Gibbs equation and the flux of η depend on the scalar invariants of the system which are formed of nonconserved variables as well as the conserved ones. The scalars of the theory are expanded around the local equilibrium state with fluxes equal to zero and a systematic order criterion to approximate these series [31]. (iv) The generalized thermodynamic potential production term can depend additionally on parameters not belonging to the extended space, since the tangent space is not generated by the extended space (this constitutes the closure assumption [30]). Moreover, the production is not assumed to be necessarily a semipositive definite quantity [32,33] opening the possibility to other extra couplings between the nonconserved variables allowed in this EIT framework, in contrast with other versions of EIT [34].

The standard procedure of EIT is to compare the two expressions of the temporal derivative of η obtained; one from the Gibbs equation when the conservation equations

are introduced in it and the other from the balance equation when the flux \mathbf{J}_η together with the production σ are written as the most general vector and scalar quantities in the extended thermodynamic space (and parameters of the closure assumption), respectively, and are substituted in Eq. (1). This comparison leads to the time evolution equations for the nonconserved variables which, together with the conservation equations, constitute a full set describing the system.

The EIT has been applied with success to a lot of problems of chemical kinetics [15,34] and generalized hydrodynamics and with encouraging results to the theory of fluctuations [35], non-Newtonian fluids [36,37], thermoelectric phenomena [38], flow in porous media [39,40], the properties of fluids driven onto far from equilibrium stationary states by external gradients [41], etc.

III. THE MODEL: A VARIATIONAL APPROACH TO THE TIME EVOLUTION EQUATIONS OF A CONDUCTING VISCOUS FLUID

In a previous work we have outlined a variational principle for systems which can be described by the EIT theory [20,42]. The variational equation is

$$\delta I = \delta \int_{\Omega} \left[\rho \frac{d\eta}{dt} + \nabla \cdot \mathbf{J}_\eta - \sigma \right] dV dt = 0, \quad (2)$$

with the following conditions: (i) The variation δ is carried out on the nonconserved variables only, (ii) the tangent thermodynamic space (time and spatial derivatives) is fixed during the variation, and (iii) the conservation equations and the representation theorems for the unknown quantities in the extended space are subsidiary conditions of the generalized Gibbs equation for an entropylike function. As can be seen, the principle (2) is of the restricted type and on the line of work of Onsager [6], Rosen [43], and Gyarmati [44]. It has the same level of generality as that of Gyarmati's principle.

Now let us suppose a conducting viscous fluid which is described by the set of thermodynamic variables: specific volume v , internal energy u , electron number density per mass unit c_e , heat flux \mathbf{q} , flux of electric current \mathbf{i} , traceless viscous stress tensor $\overline{\tau}$ and its trace τ , which constitute the extended thermodynamic space. Note that our choice of the extended space differs with respect to that of Goldstein and García-Colín [23]. These authors did not include either the flux of electric current or the associated electronic density in the extended space. So, they did not obtain an evolution equation for the electric flux but only for the heat flux and the viscous stress tensor. Nevertheless, there is a microscopic justification to introduce the equation of \mathbf{i} for the conducting viscous fluid [26]. We may also mention that in other references the rigid solid conductor has been treated including both charge density and flux of electric current and a Maxwell-Cattaneo-Vernotte equation for \mathbf{i} has been obtained [26,34,45].

The subsidiary equations of Eq. (2) are the balance equations:

$$\rho \frac{dv}{dt} = \nabla \cdot \mathbf{v} , \quad (3)$$

$$\rho \frac{dc}{dt^e} = -\nabla \cdot \mathbf{i} , \quad (4)$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \nabla \cdot \boldsymbol{\tau} + \rho z (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{i} \times \mathbf{B} , \quad (5)$$

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{q} - \rho p \frac{dv}{dt} - \vec{\tau} : \nabla \mathbf{v} + \mathbf{i} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \tau \nabla \cdot \mathbf{v} , \quad (6)$$

where we have included the effect of the complete Lorentz force, z is the charge per mass unit, \mathbf{v} is the velocity, p the pressure, and \mathbf{E} and \mathbf{B} are the electric and magnetic fields, respectively. The generalized Gibbs equation for an entropylike function is then

$$\begin{aligned} \rho \frac{d\eta}{dt} = & \beta_1 \frac{du}{dt} + \beta_2 \frac{dv}{dt} + \beta_3 \frac{dc}{dt^e} + \beta_4 \cdot \frac{d\mathbf{q}}{dt} \\ & + \vec{\beta}_5 : \frac{d\vec{\tau}}{dt} + \beta_6 \cdot \frac{d\mathbf{i}}{dt} + \beta_7 \frac{d\tau}{dt} . \end{aligned} \quad (7)$$

Here the coefficients β_i are quantities which depend on the conserved densities and the scalar invariants I_i , which in turn are constructed with the nonconserved variables of the extended space. Explicitly, the scalar, vector, and order two tensor quantities appearing in Eq. (7) are given by

$$\begin{aligned} \beta_1 = & \beta_1(u, v, c_e, I_i) , \quad \beta_2 = \beta_2(u, v, c_e, I_i) , \\ \beta_3 = & \beta_3(u, v, c_e, I_i) , \quad \beta_7 = \beta_7(u, v, c_e, I_i) , \\ \beta_4 = & \beta_{41}\mathbf{q} + \beta_{42}\mathbf{i} + \beta_{43}\vec{\tau} \cdot \mathbf{q} + \beta_{44}\vec{\tau} \cdot \mathbf{i} , \\ \beta_6 = & \beta_{61}\mathbf{q} + \beta_{62}\mathbf{i} + \beta_{63}\vec{\tau} \cdot \mathbf{q} + \beta_{64}\vec{\tau} \cdot \mathbf{i} , \\ \vec{\beta}_5 = & \beta_{51}\vec{\tau} + \beta_{52}\mathbf{q}\mathbf{q} + \beta_{53}\mathbf{i}\mathbf{i} + \beta_{54}\mathbf{i}\mathbf{q} + \beta_{55}\vec{\tau} \cdot \vec{\tau} \\ & + \beta_{56}\vec{\tau} \cdot \mathbf{i} + \beta_{57}\mathbf{q}\vec{\tau} \cdot \mathbf{q} + \beta_{58}\mathbf{q}\vec{\tau} \cdot \mathbf{i} , \\ \mathbf{J}_\eta = & \alpha_1\mathbf{q} + \alpha_2\mathbf{i} + \alpha_3\vec{\tau}\mathbf{q} + \alpha_4\vec{\tau} \cdot \mathbf{i} , \end{aligned} \quad (8)$$

where $\beta_{ij} = \beta_{ij}(u, \varphi, c_e, I)$. Since the tangent space is not fully spanned by the extended thermodynamic space, the production σ may depend on the scalar invariants and other parameters [30] which are described as

$$\sigma = \sigma(u, v, c_e, I_i, \text{parameters}) . \quad (9)$$

The scalar invariants are taken here as

$$\begin{aligned} I_1 = & \tau , \quad I_2 = \mathbf{q} \cdot \mathbf{q} , \quad I_3 = \mathbf{i} \cdot \mathbf{i} , \\ I_4 = & \mathbf{q} \cdot \mathbf{i} \quad I_5 = \text{tr}(\vec{\tau} \cdot \vec{\tau}) , \quad I_6 = \text{tr}(\vec{\tau} \cdot \vec{\tau} \cdot \vec{\tau}) , \\ I_7 = & \mathbf{q} \cdot \vec{\tau} \cdot \mathbf{q} , \quad I_8 = \mathbf{i} \cdot \vec{\tau} \cdot \mathbf{i} , \quad I_9 = \mathbf{q} \cdot \vec{\tau} \cdot \mathbf{i} , \\ I_{10} = & \mathbf{q} \cdot (\vec{\tau} \cdot \vec{\tau}) \cdot \mathbf{q} , \quad I_{11} = \mathbf{i} \cdot (\vec{\tau} \cdot \vec{\tau}) \cdot \mathbf{i} , \\ I_{12} = & \mathbf{q} \cdot (\vec{\tau} \cdot \vec{\tau}) \cdot \mathbf{i} , \end{aligned} \quad (10)$$

while the additional parameters for σ are

$$\begin{aligned} p_i = & \nabla \cdot \mathbf{v} , \quad \vec{p}_2 = (\nabla \mathbf{v})^s , \quad \vec{p}_3 = (\nabla \mathbf{v})^a , \\ p_4 = & \mathbf{E} + \mathbf{v} \times \mathbf{B} , \quad p_5 = \mathbf{i} \times \mathbf{B} , \end{aligned} \quad (11)$$

where $()^s$ and $()^a$ are the symmetric and antisymmetric parts of the tensor $\nabla \mathbf{v}$, respectively. It is convenient to remark that the choice of the parameters p_i is not made in a direct way, but rather by observing the relevant variables for the description of the system appearing in the balance equations [Eqs. (3)–(6)]. These parameters originate time and spatial inhomogeneities and for this reason they may be considered the generalized thermodynamic forces. We included the electric field \mathbf{E} in the fourth parameter in Eqs. (11) in order to consider the complete Lorentz force and do not restrict ourselves to the magnetohydrodynamics (MHD) approximation here.

Introducing Eqs. (3)–(7) in variational equation we arrive at

$$\begin{aligned} \delta \int \left[-\beta_1 \nabla \cdot \mathbf{q} - \beta_1 \rho \nabla \cdot \mathbf{v} - \beta_1 \tau : \nabla \mathbf{v} + \beta_1 \mathbf{i} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \beta_1 \tau \nabla \cdot \mathbf{v} + \beta_2 \nabla \cdot \mathbf{v} - \beta_3 \nabla \cdot \mathbf{i} - \beta_4 \cdot \frac{d\mathbf{q}}{dt} \right. \\ \left. + \vec{\beta}_5 : \frac{d\vec{\tau}}{dt} + \beta_6 \cdot \frac{d\mathbf{i}}{dt} + \beta_7 \frac{d\tau}{dt} + \alpha_1 \nabla \cdot \mathbf{q} + \mathbf{q} \cdot \nabla \alpha_1 + \alpha_2 \nabla \cdot \mathbf{i} + \mathbf{i} \cdot \nabla \alpha_2 + \alpha_3 \nabla \cdot (\vec{\tau} \cdot \mathbf{q}) + (\vec{\tau} \cdot \mathbf{q}) \cdot \nabla \alpha_3 + \alpha_4 \nabla \cdot (\vec{\tau} \cdot \mathbf{i}) \right. \\ \left. + (\vec{\tau} \cdot \mathbf{i}) \cdot \nabla \alpha_4 - \sigma \right] dV dt . \quad (12) \end{aligned}$$

The variation process leads to a set of coupled time evolution equations for the fluxes which do not involve any extra approximation. We omit them here because they are not useful for this discussion. However, they permit an overview of the formal structure of the time evolution equations in EIT. Now we approximate Eq. (12) by means of the order criterion of nonconserved thermo-

dynamic variables [31] where the order of an equation is determined by the nonequilibrium thermodynamic variables alone, no matter which specific operator is acting on them. So, if the functional in Eq. (2) is to be approximated to order two, the coefficient β_2 must also be developed to order two, while the other β 's must be expanded to order one:

$$\begin{aligned}
\beta_1 &= \beta_{10} + \beta_{11}\tau + O(2), \\
\beta_2 &= \beta_{20} + \beta_{21}\tau + \beta_{22}\mathbf{q}\cdot\mathbf{q} + \beta_{23}\mathbf{i}\cdot\mathbf{i} + \beta_{24}\mathbf{q}\cdot\mathbf{i} + \beta_{25}\vec{\tau}\cdot\vec{\tau} + O(3), \\
\beta_3 &= \beta_{30} + \beta_{31}\tau + O(2), \quad \beta_4 = \beta_{41}\mathbf{q} + \beta_{42}\mathbf{i} + O(2), \\
\vec{\beta}_5 &= \beta_{51}\vec{\tau} + O(2), \quad \beta_6 = \beta_{61}\mathbf{i} + \beta_{62}\mathbf{q} + O(2), \\
\beta_7 &= \beta_{70} + \beta_{71}\tau + O(2), \quad \alpha_1 = \alpha_{10} + \alpha_{11}\tau + O(2), \\
\alpha_2 &= \alpha_{20} + \alpha_{21}\tau + O(2), \quad \alpha_3 = \alpha_{30} + O(1), \\
\alpha_4 &= \alpha_{40} + O(1).
\end{aligned} \tag{13}$$

Here, the α_{ij} and the β_{ij} are functions of u , v , and c_e only. Since EIT must reduce to linear irreversible thermodynamics (LIT) to first order, it is required that $\beta_{11} = \beta_{21} = \beta_{31} = \beta_{70} = 0$, $\alpha_{10} = T^{-1}$, $\alpha_{20} = T^{-1}\mu_e$. In this way the two first generalized equations of the state are the state equations of LIT. T and μ_e are the temperature and the electrochemical potential at local equilibrium, respectively.

On the other hand, the production of the entropylike function to order two is

$$\begin{aligned}
\sigma &= \sigma_0\tau^2 + \sigma_1\mathbf{q}\cdot\mathbf{q} + \sigma_2\mathbf{i}\cdot\mathbf{i} + \sigma_3\vec{\tau}\cdot\vec{\tau} + \sigma_4\mathbf{p}_5\cdot\mathbf{q} \\
&+ \sigma_5(\vec{p}_2\cdot\mathbf{q})\cdot\mathbf{q} + \sigma_6(\vec{p}_3\cdot\mathbf{q})\cdot\mathbf{q} + \sigma_7\mathbf{p}_5\cdot\mathbf{i} + \sigma_8(\vec{p}_2\cdot\mathbf{i})\cdot\mathbf{i} \\
&+ \sigma_9(\vec{p}_3\cdot\mathbf{i})\cdot\mathbf{i} + \sigma_{10}(\vec{\tau}\cdot\vec{p}_2)\cdot\vec{\tau} + \sigma_{11}(\vec{\tau}\cdot\vec{p}_3)\cdot\vec{\tau} \\
&+ \sigma_{12}(\vec{\tau}\cdot\mathbf{q})\cdot\mathbf{p}_4 + \sigma_{13}(\vec{\tau}\cdot\mathbf{i})\cdot\mathbf{p}_4 + \sigma_{14}p_1\tau^2.
\end{aligned} \tag{14}$$

By substitution of Eqs. (13) and (14) in Eq. (12) and assuming both α_{30} and α_{40} constants, we obtain the next set of time evolution equations for the fluxes of the system:

$$\begin{aligned}
\frac{d\mathbf{q}}{dt} &= \lambda_1\nabla T^{-1} + \lambda_2\nabla(T^{-1}\mu_e) + \lambda_3\mathbf{q} + \lambda_4(\nabla\cdot\mathbf{v})\mathbf{q} \\
&+ \lambda_5(\nabla\mathbf{v})^s\cdot\mathbf{q} + \lambda_6(\nabla\mathbf{v})^a\cdot\mathbf{q} + \lambda_7T^{-1}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \\
&+ \lambda_8\mathbf{i}\times\mathbf{B} + \lambda_9\mathbf{i} + \lambda_{10}(\nabla\cdot\mathbf{v})\mathbf{i} + \lambda_{11}(\nabla\mathbf{v})^s\cdot\mathbf{i} \\
&+ \lambda_{12}(\nabla\mathbf{v})^a\cdot\mathbf{i} + \lambda_{13}\nabla\cdot\vec{\tau} + \lambda_{14}\vec{\tau}\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B}),
\end{aligned} \tag{15a}$$

where $\lambda_1 = -k/\beta_{61}$, $k = (\beta_{41}/\beta_{61} - \beta_{42}/\beta_{62})^{-1}$, $\lambda_2 = k/\beta_{62}$, $\lambda_3 = \frac{2k\sigma_1}{\beta_{61}}$, etc.

$$\begin{aligned}
\frac{d\mathbf{i}}{dt} &= \gamma_1T^{-1}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \gamma_2\mathbf{i}\times\mathbf{B} + \gamma_3\mathbf{i} + \gamma_4(\nabla\cdot\mathbf{v})\mathbf{i} \\
&+ \gamma_5(\nabla\mathbf{v})^s\cdot\mathbf{i} + \gamma_6(\nabla\mathbf{v})^a\cdot\mathbf{i} + \gamma_7\nabla T^{-1} + \gamma_8\nabla(T^{-1}\mu_e) \\
&+ \gamma_9\mathbf{q} + \gamma_{10}(\nabla\cdot\mathbf{v})\mathbf{q} + \gamma_{11}(\nabla\mathbf{v})^s\cdot\mathbf{q} + \gamma_{12}(\nabla\mathbf{v})^a\cdot\mathbf{q} \\
&+ \gamma_{13}\nabla\cdot\tau + \gamma_{14}(\mathbf{E} + \mathbf{v}\times\mathbf{B}),
\end{aligned} \tag{15b}$$

where $\gamma_1 = k'/\beta_{42}$, $k' = (\beta_{61}/\beta_{41} - \beta_{62}/\beta_{42})^{-1}$, $\gamma_2 = k'(\sigma_4/\beta_{41} - \sigma_8/\beta_{42})$, $\gamma_3 = -2k'\sigma_2/\beta_{42}$, etc.

$$\begin{aligned}
\frac{d\vec{\tau}}{dt} &= \xi_1\vec{\tau} + \xi_2(\nabla\cdot\mathbf{v})\vec{\tau} + \xi_3\vec{\tau}\cdot\nabla\mathbf{v} + \xi_4T^{-1}\nabla\mathbf{v} + \xi_5\nabla\mathbf{q} \\
&+ \xi_6(\mathbf{E} + \mathbf{v}\times\mathbf{B})\mathbf{q} + \xi_7\nabla\mathbf{i} + \xi_8(\mathbf{E} + \mathbf{v}\times\mathbf{B})\mathbf{i},
\end{aligned} \tag{15c}$$

where $\xi_1 = 2\sigma_3/\beta_{51}$, $\xi_2 = -2\beta_{25}/\beta_{51}$, $\xi_3 = \sigma_{13}/\beta_{51}$, $\xi_4 = \beta_{10}/\beta_{51}$, etc.

$$\frac{d\tau}{dt} = \zeta_1\tau + \zeta_2T^{-1}\nabla\cdot\mathbf{v} + \zeta_3\tau\nabla\cdot\mathbf{v}, \tag{15d}$$

where $\xi_1 = 2\sigma_0/\beta_{71}$, $\xi_2 = 1/\beta_{71}$, $\xi_3 = \sigma_{21}/\beta_{71}$.

The set (15) together with Eqs. (3)–(6) and Maxwell's equations close the set of equations to second order required to describe the conducting viscous fluid in a time scale for which the fluxes have not relaxed. They can be solved for given initial and boundary conditions whenever the coefficients λ_i , γ_i , ξ_i , and ζ_i are known. Explicit expressions for the last may be obtained from kinetic theory or experimental information. We do the first by comparing Eqs. (15) with those of Yang [24] and Spitzer [25]. Yang arrived at time evolution equations for both \mathbf{q} and $\vec{\tau}$ fluxes and Spitzer for the current flux \mathbf{i} . With respect to the first we find the next identification:

$$\begin{aligned}
\lambda_1 &= -\frac{5}{2}pT^2, \quad \lambda_3 = -\frac{2}{3}\frac{p}{\eta_v}, \quad \lambda_4 = -\frac{7}{5}, \\
\lambda_5 &= -\frac{7}{5}, \quad \lambda_6 = -\frac{2}{5}, \quad \lambda_7 = -\frac{5}{2}\frac{c_e}{\rho}pT, \\
\lambda_8 &= -\frac{5}{2}\frac{p}{\rho}, \quad \lambda_{13} = \frac{5}{2}\frac{p}{\rho}; \\
\xi_1 &= -\frac{p}{\eta_v}, \quad \xi_2 = -1, \quad \xi_3 = -2, \\
\xi_4 &= -2pT, \quad \xi_5 = -\frac{4}{5}, \quad \xi_8 = 2,
\end{aligned} \tag{16}$$

where η_v is the shear viscosity. This identification coincides with that of Cuevas [46]. The comparison with Spitzer's work gives

$$\begin{aligned}
\gamma_1 &= T(m_i m_e c^2 / Z\rho e^2)^{-1}, \\
\gamma_2 &= (Zm_e - m_i)(m_i m_e c^2 / Z\rho e^2)^{-1}, \\
\gamma_3 &= -r(m_i m_e c^2 / Z\rho e^2)^{-1}.
\end{aligned} \tag{17}$$

m_e and m_i are the electron and ion masses, respectively, Z the ionic charge, and r the resistivity.

We can recognize in $\lambda_8\mathbf{i}\times\mathbf{B}$ of Eq. (15a) an additional contribution to the heat flux which is a product of the interaction between the magnetic field and the electric flux, known as the Ettingshausen effect [47]. This term will lead to an additional term in the linear Fourier law. Also the same term in Eq. (15b) is associated with the Hall effect and it describes an electric flux perpendicular to the magnetic field.

If we make the Yang-Spitzer approximation then Eqs. (15) are an equivalent set of their kinetic equations, therefore in this approximation the remainder terms (not necessarily their respective coefficient) must be negligible. Otherwise, the remainder terms may appear for systems which do not satisfy the MHD approximation. So, we have to keep in mind the complete set of coefficients in Eqs. (15).

To stress that the comparison of our results to second order is in agreement with those stemming out of kinetic theory as derived in Yang's work let us make two additional comments: (1) second order in the Gibbs equation for the generalized thermodynamical potential η implies first order in the time evolution equation for the two non-conserved variables \mathbf{q} and τ of the system [Eqs. (15a) and (15c)], so the second-order terms in Yang's equations (4.26 and 4.27 in Ref. [24]) for the slightly ionized gas

will not appear in our equations in this approximation; however, they will show up if we go up to third order. This is due to the fact that, contrary to the systematic expansion implied in our ordering scheme, the moment method chooses to approximate the full distribution function by truncating velocity moments of higher order than those involved in the kinetic theory expressions for \mathbf{q} and $\overline{\mathbf{v}}$, which clearly is not systematic. Let us further point out that all the remainder terms in Yang's equations are present in Eqs. (15a) and (15c). (2) The terms in the flux of electric current \mathbf{i} appear when we considered this variable in the extended thermodynamic space and, in any case, the fact that they are missing in Yang's equations means that either they are negligible for a slightly ionized gas or they arise from the higher moments that have been truncated in the 13-moment approximation. Finally, those terms may have significance in systems which are not subject to these restrictions.

IV. CONCLUDING REMARKS

In this paper it is shown how the time evolution equations to second order for the dissipative fluxes in a heat- and electricity-conducting viscous fluid can be derived from a concise variational formalism. The results include those of Cuevas [46] and those of Goldstein and García-Colín [23]. If we impose the condition that the gas be a nonconducting fluid the equations of del Río and López de Haro [31] are obtained. Similarly, we can recover the results of Llebot, Jou, and Casas-Vázquez [38] constraining our equations to a rigid electrical conductor. The results of Jou, Ferrer, and Llebot [26] are not explicitly included in the approximation used here, but it is clear that the method can reproduce them, as in their paper, to order four.

From a formal point of view the principle presented here differs from other variational formulations in various aspects. The functional is defined in terms of general quantities concerning the outstanding variables of the system, we do not make any assumption about the nature of the state and phenomenological equations, i.e., about their form or properties. In this sense, the principle has the same level of generality as those of Onsager [6] and Prigogine [7], and moreover it attempts to involve systems beyond local equilibrium. In spite of the resemblance of the principles functional to that of Gyarmati [15,44], mention must be made that we do not use the

concept of dissipative potential in its definition and this fact constitutes an important difference between the two formulations. We introduce directly the production of the entropylike function η in the functional. Furthermore, we consider the fluxes as thermodynamic variables in contrast with Gyarmati's approach which includes the time derivatives of the conserved quantities as independent variables.

On the other hand, we develop the coefficients in Taylor's series around local equilibrium, but this is a possibility given by the formal development of the EIT theory. The principle, as it is shown here, inherits the EIT's capacity to develop the series to higher orders. It also offers the framework to consider nonanalytic expressions for the equations of state off local equilibrium and other phenomenological coefficients, which in the standard procedure have not been possible to treat.

The identification of the phenomenological coefficients is beyond that of Goldstein and García-Colín [23] and Cuevas [46] because it has been possible to compare the time evolution equation for the flux \mathbf{i} with the results of kinetic theory. Such an equation is lacking in Goldstein and García-Colín's paper. It must be pointed out that we have had to use the closure assumption [30] in order to make an exact comparison between the macroscopic equations and the kinetic formulation of the ionized gas.

Let us make some further remarks to conclude. First, the variational principle may contribute to the study of the consequences which EIT can introduce on the time description of the electromagnetic field (as is known, the extremum of a scalar quantity does not depend on the coordinate system, so variational principles provide an invariant description of phenomena in front of changes in the frame of reference). Second, it may provide the basis for fruitful comparisons among the extended irreversible thermodynamics used here and other extended versions for nonequilibrium processes.

ACKNOWLEDGMENTS

We wish to thank Dr. Julia Tagüena, Dr. Mariano López de Haro, and Dr. Rosalío Rodríguez from Instituto de Investigaciones en Materiales de the Universidad Nacional Autónoma de México for helpful comments and suggestions.

-
- [1] W. Thomson, Cambridge Dublin Math. J. **4**, 90 (1849).
 - [2] M. D. Krustal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).
 - [3] J. M. Greene and E. T. Karlson, Phys. Fluids **12**, 561 (1969).
 - [4] N. C. Wenger, J. Fluid Mech. **43**, 211 (1970).
 - [5] T. S. Lundgren, Phys. Fluids **6**, 898 (1963).
 - [6] L. Onsager, Phys. Rev. **37**, 405 (1931).
 - [7] I. Prigogine, Acad. R. Belg. Class. Sci. Mem. Collect. **40** **31**, 600 (1945).
 - [8] P. Glansdorff and I. Prigogine, Physica **30**, 351 (1964).
 - [9] P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations* (Wiley, London, 1971).
 - [10] E. Hameiri and A. Bhattacharjee, Phys. Rev. A **35**, 768 (1987).
 - [11] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).
 - [12] S. N. Rasband, G. W. Mason, and P. L. Matheson, Phys. Rev. A **38**, 5294 (1988).
 - [13] G. Lebon and P. C. Dauby, Phys. Rev. A **42**, 4710 (1990).
 - [14] D. K. Bhattacharya, Ann. Phys. (Leipzig) **39**, 325 (1982).
 - [15] R. E. Nettleton, J. Phys. A **19**, L295 (1986).
 - [16] S. Sieniutycz, J. Non-Equilib. Thermodyn. **9**, 61 (1984).
 - [17] S. Sieniutycz, Appl. Sci. Res. **42**, 211 (1985).
 - [18] S. Sieniutycz, Chem. Eng. Sci. **42**, 2697 (1987).
 - [19] B. C. Eu, Ann. Phys. (N.Y.) **140**, 341 (1982).

- [20] F. Vázquez and J. A. del Río, *Rev. Mex. Fis.* **36**, 71 (1990).
- [21] L. S. García-Colín, *Rev. Mex. Fis.* **34**, 344 (1988).
- [22] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 953; P. Rosen, *J. Appl. Phys.* **25**, 336 (1954).
- [23] P. Goldstein and L. S. García-Colín, *Rev. Mex. Fis.* **35**, 367 (1989).
- [24] H. T. Yang, *Phys. Fluids* **5**, 1580 (1962).
- [25] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962).
- [26] D. Jou, M. Ferrer, and J. E. Llebot, *J. Phys. A* **21**, L1039 (1988).
- [27] A. R. Vasconcelos, R. Luzzi, and L. S. García-Colín, *Phys. Rev. A* **12**, 6622 (1991).
- [28] I. Müller, *Z. Phys.* **198**, 329 (1967).
- [29] L. S. García-Colín, M. López de Haro, R. F. Rodríguez, D. Jou, and J. Casas-Vázquez, *J. Stat. Phys.* **37**, 465 (1984).
- [30] R. F. Rodríguez, and M. López de Haro, *J. Non-Equilib. Thermodyn.* **14**, 37 (1989).
- [31] J. A. del Río and M. López de Haro, *J. Non-Equilib. Thermodyn.* **15**, 59 (1990).
- [32] L. S. García-Colín and G. L. Fuentes y Martínez, *J. Stat. Phys.* **29**, 387 (1982).
- [33] L. S. García-Colín and M. López de Haro, *J. Non-Equilib. Thermodyn.* **7**, 95 (1982).
- [34] D. Lou, J. Casas-Vázquez, and G. Lebon, *Rep. Prog. Phys.* **51**, 1105 (1988).
- [35] D. Jou, C. Pérez García, L. S. García-Colín, M. López de Haro, and R. F. Rodríguez, *Phys. Rev. A* **31**, 2502 (1986).
- [36] M. López de Haro, L. F. del Castillo, and R. F. Rodríguez, *Rheol. Acta* **25**, 207 (1986).
- [37] G. Lebon, C. Pérez García, and J. Casas-Vázquez, *Physica A* **137**, 531 (1986).
- [38] J. E. Llebot, D. Jou, and J. Casas-Vázquez, *Physica* **121A**, 552 (1983).
- [39] G. Lebon, and A. Clout, *Int. J. Heat Mass Transfer* **29**, 381 (1986).
- [40] J. A. del Río and M. López de Haro, in *Physical Phenomena in Granular Materials*, edited by G. D. Cody, T. H. Geballe, and Ping Sheng, MRS Symposia Proceedings No. 195 (Materials Research Society, Pittsburgh, 1990), p. 283.
- [41] R. M. Velasco and L. S. García-Colín, *J. Phys. A* **15**, 1675 (1982).
- [42] J. A. del Río, M. López de Haro, and F. Vázquez, *J. Non-Equilib. Thermodyn.* **17**, 67 (1992).
- [43] P. Rosen, *J. Chem. Phys.* **21**, 1220 (1953).
- [44] I. Gyarmati, *Non-equilibrium Thermodynamics* (Springer-Verlag, New York, 1970).
- [45] D. Jou and J. E. Llebot, *J. Phys. A* **13**, L47 (1980).
- [46] S. R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics* (Dover, New York, 1984).
- [47] M. Lax, *Rev. Mod. Phys.* **32**, 25 (1960).